

Chapter 5 Initial-value problem

Consider $\frac{dy}{dt} = f(t, y)$, $a \leq t \leq b$, $y(a) = \alpha$

Expand $y(t_{i+1}) = y(t_i) + y'(t_i)(t_{i+1} - t_i) + \frac{1}{2}y''(t_i)(t_{i+1} - t_i)^2 + \frac{1}{3!}y'''(t_i)(t_{i+1} - t_i)^3 + \dots$

$$y(t_{i+1}) = y(t_i) + f[t_i, y(t_i)](t_{i+1} - t_i) + \frac{1}{2} \frac{d}{dt} f[t_i, y(t_i)](t_{i+1} - t_i)^2 + \dots$$

Euler's method

$$h = t_{i+1} - t_i, \quad y(t_{i+1}) = y(t_i) + f[t_i, y(t_i)]h$$

Taylor method of order n

$$y(t_{i+1}) = y(t_i) + hT^{(n)}[t_i, y(t_i)]$$

where

$$T^{(n)}[t_i, y(t_i)] = f[t_i, y(t_i)] + \frac{1}{2!} \frac{df}{dt} [t_i, y(t_i)]h + \dots + \frac{1}{n!} \frac{d^{n-1}f}{dt^{n-1}} h^{n-1}$$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} = \frac{\partial f}{\partial t} + f \frac{\partial f}{\partial y}$$

$$\begin{aligned} \frac{d^2 f}{dt^2} &= \frac{d}{dt} \left(\frac{\partial f}{\partial t} + f \frac{\partial f}{\partial y} \right) = \left[\frac{\partial}{\partial t} \left(\frac{\partial f}{\partial t} \right) + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial t} \right) \frac{\partial y}{\partial t} \right] + \left[\frac{\partial}{\partial t} \left(f \frac{\partial f}{\partial y} \right) + \frac{\partial}{\partial y} \left(f \frac{\partial f}{\partial y} \right) \frac{\partial y}{\partial t} \right] \\ &= \frac{\partial^2 f}{\partial t^2} + 2f \frac{\partial^2 f}{\partial y \partial t} + \frac{\partial f}{\partial t} \frac{\partial f}{\partial y} + f^2 \frac{\partial^2 f}{\partial y^2} + f \left(\frac{\partial f}{\partial y} \right)^2 \end{aligned}$$

Order 2:

$$T^{(2)} = f + \frac{h}{2!} \frac{df}{dt} = f + \frac{h}{2} \left[\frac{\partial f}{\partial t} + f \frac{\partial f}{\partial y} \right]$$

Order 3:

$$T^{(3)} = f + \frac{1}{2} \frac{df}{dt} h + \frac{1}{6} \frac{d^2 f}{dt^2} h^2.$$

$$\frac{d^2 f}{dt^2} = \frac{\partial^2 f}{\partial t^2} + 2f \frac{\partial^2 f}{\partial t \partial y} + \frac{\partial f}{\partial t} \frac{\partial f}{\partial y} + f \left(\frac{\partial f}{\partial y} \right)^2 + f^2 \frac{\partial^2 f}{\partial y^2}$$

Order 4

$$T^{(4)} = f + \frac{1}{2} \frac{df}{dt} h + \frac{1}{6} \frac{d^2 f}{dt^2} h^2 + \frac{1}{24} \frac{d^3 f}{dt^3} h^3.$$

$$\frac{d^3 f}{dt^3} = \frac{\partial^3 f}{\partial t^3} + 3f \frac{\partial^3 f}{\partial t^2 \partial y} + 3 \frac{\partial f}{\partial t} \frac{\partial^2 f}{\partial t \partial y} + 5f \frac{\partial f}{\partial y} \frac{\partial^2 f}{\partial t \partial y} + 3f^2 \frac{\partial^3 f}{\partial t \partial y^2} + 4f^2 \frac{\partial f}{\partial y} \frac{\partial^2 f}{\partial y^2}$$

$$+\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial t^2}+\frac{\partial f}{\partial t}\left(\frac{\partial f}{\partial y}\right)^2+2f\frac{\partial f}{\partial t}\frac{\partial^2 f}{\partial y^2}+f\frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial y}\right)^2+f^3\frac{\partial^3 f}{\partial y^3}.$$

Runge-Kutta method

A. Order 2

Comparing

$$y(t_{i+1}) = y(t_i) + hf[t_i + \alpha, y(t_i) + \beta]$$

with

$$y(t_{i+1}) = y(t_i) + hT^{(2)}[t_i, y(t_i)],$$

where

$$f(t + \alpha, y + \beta) = f(t, y) + \left(\alpha \frac{\partial f}{\partial t} + \beta \frac{\partial f}{\partial y}\right)$$

$$T^{(2)} = f + \frac{h}{2!} \frac{df}{dt} = f + \frac{h}{2} \left[\frac{\partial f}{\partial t} + f \frac{\partial f}{\partial y} \right]$$

Yields

$$\alpha = \frac{1}{2}h, \quad \beta = \frac{1}{2}hf.$$

$$\text{So, } y_{i+1} = y_i + hf\left[t_i + \frac{h}{2}, y_i + \frac{h}{2}f(t_i, y_i)\right].$$

Order 3

$$y_{i+1} = y_i + a_1 f(t_i, y_i) + a_2 f[t_i + \alpha_1, y_i + \beta_1 f(t_i, y_i)]$$

$$+ a_3 f\{t_i + \alpha_2, y_i + \beta_2 f[t_i + \alpha_3, y_i + \beta_3 f(t_i, y_i)]\}$$

B. Order 4

$$y_{i+1} = y_i + a_1 f(t_i, y_i) + a_2 f[t_i + \alpha_1, y_i + \beta_1 f(t_i, y_i)]$$

$$+ a_3 f\{t_i + \alpha_2, y_i + \beta_2 f[t_i + \alpha_2, y_i + \beta_3 f(t_i, y_i)]\}$$

$$+ a_4 f\left(t_i + \alpha_3, y_i + \beta_4 f\{t_i + \alpha_3, y_i + \beta_5 f[t_i + \alpha_4, y_i + \beta_6 f(t_i, y_i)]\}\right)$$

$$w_0 = \alpha, \quad k_1 = hf(t_i, w_i), \quad k_2 = hf\left(t_i + \frac{h}{2}, w_i + \frac{k_1}{2}\right), \quad k_3 = hf\left(t_i + \frac{h}{2}, w_i + \frac{k_2}{2}\right),$$

$$k_4 = hf(t_{i+1}, w_i + k_3), \quad w_{i+1} = w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) + O(h^5)$$

System of Equations

$$\frac{dy_i}{dt} = f_i(t, y_1, \dots, y_n), \quad y_i(a) = \alpha_i$$

$$k_{1,i} = hf_i(t_j, w_{1,j}, \dots, w_{n,j}), \quad k_{2,i} = hf_i(t_j + \frac{h}{2}, w_{1,j} + \frac{k_{1,1}}{2}, \dots, w_{n,j} + \frac{k_{1,n}}{2}),$$

$$k_{3,i} = hf_i(t_j + \frac{h}{2}, w_{1,j} + \frac{k_{2,1}}{2}, \dots, w_{n,j} + \frac{k_{2,n}}{2}), \quad k_{4,i} = hf_i(t_j + h, w_{1,j} + \frac{k_{2,1}}{2}, \dots, w_{n,j} + \frac{k_{2,n}}{2})$$

$$w_{i,j+1} = w_{i,j} + \frac{1}{6}(k_{1,i} + 2k_{2,i} + 2k_{3,i} + k_{4,i}) + O(h^5)$$

Ex.

$$y'' - 2y' + 2y = e^{2t} \sin t, \quad 0 \leq t \leq 1, \quad y(0) = -0.4, \quad y'(0) = -0.6$$

Ans. Set $x_1 = y$ and $x_2 = y'$.

The system of equations will be $x_2' = e^{2t} \sin t + 2x_2 - 2x_1$ and $x_1' = x_2$.

The other Methods

$y(t_{i+1}) \approx w_i + \int_{t_i}^{t_{i+1}} P(t)dt$, where $P(t)$ is the interpolating polynomial of

$$(t_0, w_0), \dots, (t_i, w_i)$$

Adams-Bashforth two-step explicit method

$$w_0 = \alpha, \quad w_1 = \alpha_1,$$

$$w_{i+1} = w_i + \frac{h}{2}[3f(t_i, w_i) - f(t_{i-1}, w_{i-1})] + O(h^3)$$

Adams-Bashforth three-step explicit method

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2$$

$$w_{i+1} = w_i + \frac{h}{12}[23f(t_i, w_i) - 16f(t_{i-1}, w_{i-1}) + 5f(t_{i-2}, w_{i-2})] + O(h^4)$$

Adams-Bashforth four-step explicit method

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2, \quad w_3 = \alpha_3$$

$$w_{i+1} = w_i + \frac{h}{24}[55f(t_i, w_i) - 59f(t_{i-1}, w_{i-1}) + 37f(t_{i-2}, w_{i-2}) - 9f(t_{i-3}, w_{i-3})] + O(h^5)$$

$y(t_{i+1}) \approx w_i + \int_{t_i}^{t_{i+1}} P(t)dt$, where $P(t)$ is the interpolating polynomial of

$$(t_0, w_0), \dots, (t_i, w_i) \quad \text{and} \quad (t_{i+1}, f(t_{i+1}, y(t_{i+1}))).$$

Adams-Moulton two-step explicit method

$$w_0 = \alpha, \quad w_1 = \alpha_1,$$

$$w_{i+1} = w_i + \frac{h}{12}[5f(t_{i+1}, w_{i+1}) + 8f(t_i, w_i) - f(t_{i-1}, w_{i-1})] + O(h^4)$$

Adams- Moulton three-step explicit method

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2$$

$$w_{i+1} = w_i + \frac{h}{24}[9f(t_{i+1}, w_{i+1}) + 19f(t_i, w_i) - 5f(t_{i-1}, w_{i-1}) + f(t_{i-2}, w_{i-2})] + O(h^5)$$

Adams- Moulton four-step explicit method

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2, \quad w_3 = \alpha_3$$

$$w_{i+1} = w_i + \frac{h}{720}[251f(t_{i+1}, w_{i+1}) + 646f(t_i, w_i) - 246f(t_{i-1}, w_{i-1}) + 106f(t_{i-2}, w_{i-2}) - 19f(t_{i-3}, w_{i-3})] + O(h^6)$$

Milne's method

$$w_{i+1} = w_{i-3} + \frac{4h}{3}[2f(t_i, w_i) - f(t_{i-1}, w_{i-1}) + 2f(t_{i-2}, w_{i-2})] + O(h^5)$$

Simpson's method

$$w_{i+1} = w_{i-1} + \frac{h}{3}[f(t_{i+1}, w_{i+1}) + 4f(t_i, w_i) + f(t_{i-1}, w_{i-1})] + O(h^5)$$

Runge-Kutta-Fehlberg method (order of 5)

$$w_{i+1} = w_i + \frac{16}{135}k_1 + \frac{6656}{12825}k_3 + \frac{28561}{56430}k_4 - \frac{9}{50}k_5 + \frac{2}{55}k_6$$

$$k_1 = hf(t_i, w_i), \quad k_2 = hf(t_i + \frac{h}{4}, w_i + \frac{k_1}{4}), \quad k_3 = hf(t_i + \frac{3h}{8}, w_i + \frac{3k_1}{32} + \frac{9k_2}{32}),$$

$$k_4 = hf(t_i + \frac{12h}{13}, w_i + \frac{1932k_1}{2197} - \frac{7200k_2}{2197} + \frac{7296k_3}{2197}),$$

$$k_5 = hf(t_i + h, w_i + \frac{439k_1}{216} - 8k_2 + \frac{3680k_3}{513} - \frac{845k_4}{4104})$$

$$k_6 = hf(t_i + \frac{1}{2}h, w_i - \frac{8k_1}{27} + 2k_2 - \frac{3544k_3}{2565} + \frac{1859k_4}{4104} - \frac{11k_5}{40})$$

HW5

1. The initial-value problem

$$y' = 1 + (y/t) + (y/t)^2, \quad 1 \leq t \leq 2, \quad y(1) = 0 \quad \text{has the exact}$$

$$\text{solution } y(t) = t \tan(\ln t).$$

a. Use Euler's method with $h = 0.1$ to approximate the solution, and

compare it with the actual values of y .

- b. Use Taylor's method of order 2 with $h = 0.1$ to approximate the solution, and compare it with the actual values of y .

2. The system of initial-value problems

$$u_1' = 9u_1 + 24u_2 + 5\cos t - \frac{1}{3}\sin t, \quad u_1(0) = \frac{4}{3},$$

$$u_2' = -24u_1 - 52u_2 - 9\cos t + \frac{1}{3}\sin t, \quad u_2(0) = \frac{2}{3},$$

has the unique solution

$$u_1 = 2e^{-3t} - e^{-39t} + \frac{1}{3}\cos t, \quad u_2 = -e^{-3t} + 2e^{-39t} - \frac{1}{3}\cos t.$$

Try $h = 0.05$ and $h = 0.1$ in Runge-Kutta method, and compare their results with the exact value.